Q1

[6

$$i = \sqrt{-1} \Rightarrow i^2 = -1$$

$$Z = a + bi$$

$a = Re(z)$ $\quad$ $b = Im(z)$

Note: Though z is a complex number, $Re(z)$ and $Im(z)$ are just 'normal' real numbers.

$$z = a + bi \qquad z^* = a - bi$$

z^* is the complex conjugate of z

Note: $z \times z^*$ is always a real number!

$$zz^* = (a+bi)(a-bi) = a^2 - b^2 i^2 = a^2 + b^2$$

(i) $z_2 - z_1 = (-3+5i) - (1-2i) = -4 + 7i$

$$\boxed{Re(z_2 - z_1) = -4}$$

(ii) $z_1 z_2 = (1-2i)(-3+5i) = -3 + 5i + 6i - 10i^2$

$$z_1 z_2 = -3 + 5i + 6i - 10(-1) = 7 + 11i$$

$$\boxed{Im(z_1 z_2) = 11}$$

(iii) $\dfrac{z_1}{z_2} = \dfrac{1-2i}{-3+5i} \times \dfrac{-3-5i}{-3-5i}$

$$= \frac{-3-5i+6i+10i^2}{9+\cancel{15i}-\cancel{15i}-25i^2}$$

$$= \frac{-3-5i+6i-10}{9+25}$$

$$\frac{z_1}{z_2} = \frac{-13+i}{34} = -\frac{13}{34} + \frac{1}{34}i$$

$$\boxed{\left(\frac{z_1}{z_2}\right)^* = -\frac{13}{34} - \frac{1}{34}i}$$

Q2

$(2+5i)(z+2i) = -7-32i$

$(2+5i)z + (2+5i)2i = -7-32i$

$(2+5i)z + 4i + 10i^2 = -7-32i$

$(2+5i)z + 4i - 10 = -7-32i$

$(2+5i)z = 3-36i$

$z = \dfrac{3-36i}{2+5i}$

$z = \dfrac{3-36i}{2+5i} \times \dfrac{2-5i}{2-5i}$

$= \dfrac{6-15i-72i+180i^2}{4-\cancel{10i}+\cancel{10i}-25i^2}$

$= \dfrac{6-15i-72i-180}{4+25}$

$z = \dfrac{-174-87i}{29}$

$\boxed{z = -6-3i}$

Q3a

(a) Given that $z_1 = a - 6i$, $z_2 = 1 + bi$, and $z_1 z_2 = -17 - 9i$, where a and b are real numbers, find the possible values of a and b.

[4]

(b) Using your answers to part (a), write down values for c and d that will satisfy the equation

$$-(3+i)(c+di) = -17-9i$$

[2]

Important fact

Two complex numbers are equal if and only if both their real and imaginary parts are equal.

a) $z_1 z_2 = (a-6i)(1+bi) = -17-9i$

So $a + abi - 6i - 6bi^2 = -17-9i$

$a + abi - 6i - 6b(-1) = -17-9i$

$(a+6b) + (ab-6)i = -17-9i$ equal equal

This can only be true if

$a + 6b = -17$ ① ← simultaneous equations

and $ab - 6 = -9 \Rightarrow ab = -3 \Rightarrow b = -\frac{3}{a}$ ②

Substitute ② into ①:

$a + 6\left(-\frac{3}{a}\right) = -17$

$a - \frac{18}{a} = -17$

$a^2 - 18 = -17a$

$a^2 + 17a - 18 = (a+18)(a-1) = 0$

So $a = 1 \Rightarrow b = -\frac{3}{1} = -3$

or $a = -18 \Rightarrow b = -\frac{3}{-18} = \frac{1}{6}$

$\boxed{a=1,\ b=-3 \quad \text{or} \quad a=-18,\ b=\frac{1}{6}}$

Q3b

b) Using $a = -18$ and $b = \frac{1}{6}$ from part (a):

$$(-18-6i)(1+\tfrac{1}{6}i) = -17-9i$$
$$-6(3+i)(1+\tfrac{1}{6}i) = -17-9i$$
$$-(3+i)(6)(1+\tfrac{1}{6}i) = -17-9i$$
$$-(3+i)(6+i) = -17-9i$$

$$\boxed{c = 6,\ d = 1}$$

Q4

The equation will have non-real roots if the discriminant is less than zero:

$$b^2 - 4(1)(18) < 0$$
$$b^2 - 72 < 0$$
$$b^2 < 72$$
$$-\sqrt{72} < b < \sqrt{72}$$

$$\boxed{-6\sqrt{2} < b < 6\sqrt{2}}$$

Q5

Given that $-3+2i$ is one of the roots of the quadratic equation $z^2+bz+c=0$, where b and c are real constants, find the values of b and c.

[4]

Important fact

In quadratic, cubic, and quartic equations with real coefficients, non-real complex roots only occur in complex conjugate pairs.

The other root is $(-3+2i)^* = -3-2i$.

Therefore

$$z^2+bz+c = (z-(-3+2i))(z-(-3-2i))$$
$$= z^2-(-3-2i)z-(-3+2i)z+(-3+2i)(-3-2i)$$
$$= z^2+(3+\cancel{2i}+3-\cancel{2i})z+(9+\cancel{6i}-\cancel{6i}-4i^2)$$
$$= z^2+6z+(9-4(-1))$$

$$z^2+bz+c = z^2+6z+13$$

$$\boxed{b=6 \quad c=13}$$

Q6

Important fact

In quadratic, cubic, and quartic equations with real coefficients, non-real complex roots only occur in complex conjugate pairs.

Note: This fact doesn't apply here, because the z coefficient (2i) is not a real number!

Complete the square

(i) $z^2 + 2iz - 10 = (z+i)^2 - i^2 - 10$

$= (z+i)^2 - (-1) - 10$

$$\boxed{\begin{array}{c} f(z) = z^2 + 2iz - 10 = (z+i)^2 - 9 \\ (a = 1,\ b = -9) \end{array}}$$

(ii) $f(z) = 0$

$z^2 + 2iz - 10 = 0$

$(z+i)^2 - 9 = 0$

$(z+i)^2 = 9$

$z + i = \pm 3$

$z = -i \pm 3$

The solutions are

$$\boxed{z = 3 - i \quad \text{or} \quad z = -3 - i}$$

Not complex conjugates

Q7A

a) $\alpha^2 = (-1+4i)(-1+4i)$

$= 1 - 4i - 4i + 16i^2$

$= 1 - 8i - 16 = -15 - 8i$

$\alpha^3 = \alpha^2\alpha = (-15-8i)(-1+4i)$

$= 15 - 60i + 8i - 32i^2$

$= 15 - 52i + 32 = 47 - 52i$

$(47-52i) + 5(-15-8i) + 23(-1+4i) + 51$

$= (47 - 75 - 23 + 51) + (-52i - 40i + 92i)$

$= 0 + 0 = 0$

So $-1+4i$ is a root of the equation.

Q7b

(a) Show that $\alpha = -1 + 4i$ is a root of the cubic equation

$$z^3 + 5z^2 + 23z + 51 = 0$$

[3]

(b) Find the other two roots of the cubic equation in part (a), being sure to show clear algebraic working.

[3]

Important fact

In quadratic, cubic, and quartic equations with real coefficients, non-real complex roots only occur in complex conjugate pairs.

b) Another root is $(-1+4i)^* = -1-4i$

Therefore

$(z-(-1+4i))(z-(-1-4i))$

$= z^2 - (-1-4i)z - (-1+4i)z + (-1+4i)(-1-4i)$

$= z^2 + 2z + (1 - 16i^2)$

$= z^2 + 2z + 17$

is a factor of $z^3 + 5z^2 + 23z + 51$.

So

$z^3 + 5z^2 + 23z + 51 = (z^2 + 2z + 17)(z + 3) = 0$

Therefore -3 is the third root

The other two roots are

$-1-4i$ and -3

Q8

$f(z) = z^3 + z^2 + cz + d$, where c and d are real numbers.

Given that 3 and $-2 - 3i$ are roots of the equation $f(z) = 0$, find the value of c and the value of d.

[5]

Important fact

In quadratic, cubic, and quartic equations with real coefficients, non-real complex roots only occur in complex conjugate pairs.

$(-2-3i)^* = -2+3i$ is another root

Therefore

$$f(z) = (z-3)(z-(-2-3i))(z-(-2+3i))$$
$$= (z-3)(z^2+4z+(4-9i^2))$$
$$= (z-3)(z^2+4z+13)$$
$$= z^3+4z^2+13z-3z^2-12z-39$$

$$f(z) = z^3+z^2+z-39$$

$$\boxed{c = 1 \quad d = -39}$$

Q9

For a complex number z, a square root of z is a complex number $x + iy$ where x and y are real numbers and where

$$(x + iy)^2 = z$$

By expanding $(x + iy)^2$ and solving the resultant equation for x and y, determine the two square roots of the complex number $z = -21 - 20i$.

[6]

Important fact

Two complex numbers are equal if and only if both their real and imaginary parts are equal.

$z = a + bi$

$a = \mathrm{Re}(z)$ $\quad$ $b = \mathrm{Im}(z)$

Note: Though z is a complex number, $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are just 'normal' real numbers.

$$(x+iy)^2 = (x+iy)(x+iy) = x^2+xyi+xyi+i^2y^2$$
$$= (x^2-y^2)+(2xy)i = -21-20i$$

This can only be true if

$x^2-y^2 = -21$ ①

and $2xy = -20 \implies y = -\frac{10}{x}$ ② $\Big\}$ simultaneous equations

Substituting ② into ①:

$$x^2 - \left(-\frac{10}{x}\right)^2 = -21$$
$$x^2 - \frac{100}{x^2} = -21$$
$$x^4 + 21x^2 - 100 = 0 \quad \Big\}\text{ 'Hidden quadratic' in } x^2$$
$$(x^2-4)(x^2+25) = 0$$
$$x^2 = 4 \text{ or } x^2 = -25$$

But only $x^2 = 4$ has real solutions, so

$x = 2 \implies y = -\frac{10}{2} = -5$

or $x = -2 \implies y = -\frac{10}{-2} = 5$

Therefore the square roots of $-21-20i$ are

$$\boxed{2-5i \text{ and } -2+5i}$$

Note that this answer could also be given in the form $\pm(2-5i)$